XXIV.—*On the development of the disturbing Function, upon which depend the inequalities of the motions of the Planets, caused by their mutual attraction.* By James Ivory, *K.H., M.A., F.R.S., Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

Received May 30,—Read June 20, 1833.

The perturbations of the planets caused by their mutual attraction depend chiefly upon one algebraic expression, from the development of which all the inequalities of their motions are derived. This function is very complicated, and requires much labour and many tedious operations to expand it in a series of parts which can be separately computed according to the occasions of the astronomer. The progress of physical astronomy has undoubtedly been retarded by the excessive length and irksomeness attending the arithmetical calculation of the inequalities. On this subject astronomers generally and continually complain; and that their complaints are well founded, is very aptly illustrated by a paper contained in the last year's Transactions of this Society.

The disturbing function is usually expanded in parts arranged according to the powers and products of the excentricities and the inclinations of the orbits to the ecliptic; and, as these elements are always small, the resulting series decreases in every case with great rapidity. No difficulty would therefore be found in this research, if an inequality depended solely on the quantity of the coefficient of its argument in the expanded function; because the terms of the series decrease so fast, that all of them, except those of the first order, or, at most, those of the first and second orders, might be safely neglected, as producing no sensible variation in the planet's motion. But the magnitude of an inequality depends upon the length of its period, as well as upon the coefficient of its argument. When the former embraces a course of many years, the latter, although almost evanescent in the differential equation, acquires a great multiplier in the process of integration, and thus comes to have a sensible effect on the place of the planet. Such is the origin of some of the most remarkable of the planetary irregularities, and in particular, of the great equations in the mean motions of Jupiter and Saturn, the discovery of which does so much honour to the sagacity of LAPLACE. It is not, therefore, enough to calculate the terms of the first order, or of the first and second orders, in the expansion of the disturbing function. This is already done in most of the books that treat of physical astronomy with all the care and fulness which the importance of the subject demands, leaving little room for further improvement. In the present state of the theory of the planetary motions, it is requisite that the astronomer have it in his power to compute any term in the expansion of the disturbing function below the sixth order; since it has been found that there are inequalities depending upon terms of the fifth order, which have a sensible effect on the motions of some of the planets.

A research that has for its object the lessening of the difficulty attending the expansion of the disturbing function, and the bringing of that expression more under the power of the astronomer, is one of considerable interest, and of some moment to the progress of physical astronomy. But the question is not to exhibit separately every individual argument with its coefficient, which would be of little utility in practice, since, although their number be infinite, very few of them are of any account in computing the place of a planet. The end to be aimed at, is to give the function such a form that the astronomer may have it in his power to select any inequality he may wish to examine, and to compute the coefficient of its argument by an arithmetical process of moderate length. The present Paper is an attempt of this kind. The investigation comprehends every argument not passing the fifth order; but, as the formulas are regular, the method may be extended indefinitely to any order.

1. Let x, y, z, represent the rectangular coordinates of the disturbed planet, z being perpendicular to a fixed plane passing through the sun's centre, at which point the origin of the coordinates is placed: in like manner let x', y', z' be the coordinates of the disturbing planet; and put m' for the proportion of the mass of this planet to the sum of the masses of the sun and the disturbed planet; then, R denoting the disturbing function, we shall have,

$$R = m'. \left\{ \frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} - \frac{xx' + yy' + zz'}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} \right\}.$$

If r and r' denote the distances of the disturbed and disturbing planets from the sun's centre; ω, their angular distance seen at the same point; and ϱ, their rectilinear distance from one another, then,

$$\varrho = \sqrt{r^2 - 2\,r\,r'\cos\omega + r'^2}$$

$$\frac{\mathrm{R}}{m'} = \frac{1}{\varrho} - \frac{r\cos\omega}{r'^2}.$$

It appears, therefore, that the function to be expanded consists of two parts, of which $\frac{1}{\varrho}$, the reciprocal of the distance of the two planets, is common to the disturbing functions of both; but this is not the case with the other part, in which r and r' do not enter alike. For this reason it will contribute to distinctness to expand the two parts separately. But, previously, it is necessary to express $\cos\omega$ in terms of the angular motions of the planets in their respective orbits.

If υ represent the angular distance of the disturbed planet in its orbit from a fixed origin, and P the place of the node, that is, of the intersection of the orbit with the immoveable plane of $x\,y$, P being reckoned in the same plane and from the same origin as υ, then $\upsilon - \mathrm{P}$ will be the angular distance of the planet from the node. Further, if A be the celestial arc between the node and the intersection of the orbits of the two planets, the distance of the planet from the same intersection will be equal to

$$\upsilon - \mathrm{P} - \mathrm{A}.$$

If the foregoing symbols be accented and transferred with like significations to the disturbing planet, the celestial arc between that planet and the intersection of the orbits will be equal to

$$\upsilon' - \mathrm{P}' - \mathrm{A}'.$$

Now $\upsilon - \mathrm{P} - \mathrm{A}$ and $\upsilon' - \mathrm{P}' - \mathrm{A}'$ are two sides of a triangle of which the arc ω is the third side; wherefore if I be the inclination of the two orbits, we shall have

$$\begin{aligned}\cos\omega = {} & \cos(\upsilon - \mathrm{P} - \mathrm{A})\cos(\upsilon' - \mathrm{P}' - \mathrm{A}') \\ & + \cos\mathrm{I}\sin(\upsilon - \mathrm{P} - \mathrm{A})\sin(\upsilon' - \mathrm{P}' - \mathrm{A}'),\end{aligned}$$

or, which is the same thing,

$$\begin{aligned}\cos\omega = {} & \cos^2\tfrac{1}{2}\mathrm{I}\,.\cos(\upsilon - \upsilon' - \mathrm{P} + \mathrm{P}' - \mathrm{A} + \mathrm{A}') \\ & + \sin^2\tfrac{1}{2}\mathrm{I}\,.\cos(\upsilon + \upsilon' - \mathrm{P} - \mathrm{P}' - \mathrm{A} - \mathrm{A}').\end{aligned}$$

Let N be the place of the node of the disturbed planet reckoned in the immoveable plane of $x\,y$, and i the inclination of the orbit to the same plane: then, if ν represent the longitude of the intersection of the orbits of the two planets, A will be the hypothenuse of a right-angled triangle, of which i is one angle, and $\nu - \mathrm{N}$ the side adjacent to i; wherefore

$$\tan \mathrm{A} = \frac{\tan(\nu - \mathrm{N})}{\cos i};$$

and, by the usual methods,

$$\mathrm{A} = \nu - \mathrm{N} + \tan^2 \frac{i}{2} \sin 2(\nu - \mathrm{N}) + \frac{1}{2} \tan^2 \frac{i}{2} \sin 4(\nu - \mathrm{N}), \&c.;$$

and if we put

$$\xi = \tan^2 \frac{i}{2} \sin 2(\nu - \mathrm{N}) + \frac{1}{2} \tan^2 \frac{i}{2} \sin 4(\nu - \mathrm{N}), \&c.,$$

we shall have

$$\mathrm{P} + \mathrm{A} = \nu + \mathrm{P} - \mathrm{N} + \xi.$$

Let $[\mathrm{P} - \mathrm{N}]$ be the value of $\mathrm{P} - \mathrm{N}$ at some given epoch: then the differentials of N and P being $d\mathrm{N}$ and $\cos i \,.\, d\mathrm{N}$, we shall have generally,

$$\mathrm{P} - \mathrm{N} = [\mathrm{P} - \mathrm{N}] - \int d\mathrm{N} + \int \cos i \,.\, d\mathrm{N} = [\mathrm{P} - \mathrm{N}] - 2\int \sin^2 \frac{i}{2}\, d\mathrm{N},$$

the integral being taken for the time elapsed from the epoch. It thus appears that $\mathrm{P} - \mathrm{N}$ may be considered as an invariable arc, or as one subject to an almost insensible secular equation. In order to abridge expressions, let us now put

$$\alpha = [\mathrm{P} - \mathrm{N}] - 2\int \sin^2 \frac{i}{2}\, d\mathrm{N} + \xi;$$

then

$$v - \mathrm{P} - \mathrm{A} = v - \alpha - \nu.$$

If the symbols which stand for the elements of the disturbed planet be assumed, when accented, to represent the like elements of the disturbing planet, we shall have similarly,

$$\alpha' = [\mathrm{P}' - \mathrm{N}'] - 2\int \sin^2 \frac{i'}{2}\, d\mathrm{N}' + \xi',$$

$$v' - \mathrm{P}' - \mathrm{A}' = v' - \alpha' - \nu.$$

The value of cos ω will now be thus expressed,

$$\cos\omega = \cos^2\frac{1}{2}\mathrm{I}\,.\cos\left(v - v' - \alpha + \alpha'\right)$$

$$+ \sin^2\frac{1}{2}\mathrm{I}\,.\cos\left(v + v' - \alpha - \alpha' - 2\nu\right).$$

It remains to determine I, the inclination of the orbits, and ν, the longitude of their intersection. Now I is the vertical angle of a triangle of which N − N′ is the base, and i and i' the adjacent angles, one being interior and the other exterior to the triangle: wherefore

$$\cos\mathrm{I} = \cos i\cos i' + \sin i\sin i'\cos\left(\mathrm{N} - \mathrm{N}'\right).$$

In the same triangle, a perpendicular being let fall upon the base from the vertical angle, $\nu - \mathrm{N}$ and $\nu - \mathrm{N}'$ are the arcs between the perpendicular and the extremities of the base; and hence,

$$\cot\left(\nu - \frac{\mathrm{N} + \mathrm{N}'}{2}\right) = \frac{\sin\left(i - i'\right)}{\sin\left(i + i'\right)}\,.\cot\frac{\mathrm{N} - \mathrm{N}'}{2}.$$

In the method here followed, the expression of cos ω is as simple and as little troublesome in calculation, as it would be if one of the orbits were adopted for the immoveable plane.

2. Let ζ, ε, e, ϖ represent the mean motion, the epoch, the excentricity, and the place of the perihelion, of the disturbed planet at the time for which we are computing: if this time be near the epoch for which the elements have been assigned, the values at the epoch may be taken as the true values; and if a great interval has elapsed, the values at the epoch may be corrected by their secular equations. Further, put μ for the mean anomaly, σ for the equation of the centre, a for the mean distance, and $a\,(1 - s)$ for the radius vector: then,

$$v = \zeta + \varepsilon + \sigma,$$
$$\mu = \zeta + \varepsilon - \varpi,$$
$$\gamma = a\left(1 - s\right),$$

And the values of σ and s, found by the solution of Kepler's problem, are expressed by the usual series, viz.

$$\begin{aligned}\sigma = {} & e \,.\, 2\sin\mu \\ & + e^2 \,.\, \frac{5\sin 2\mu}{4} \\ & + e^3 \,.\, \frac{13\sin 3\mu - 3\sin\mu}{12} \\ & + e^4 \,.\, \frac{103\sin 4\mu - 44\sin 2\mu}{96} \\ & + e^5 \,.\, \frac{1097\sin 5\mu - 645\sin 3\mu + 50\sin\mu}{960}\end{aligned} \qquad \begin{aligned} s = {} & e\cos\mu \\ & + e^2 \,.\, \frac{\cos 2\mu - 1}{2} \\ & + e^3 \,.\, \frac{3\cos 3\mu - 3\cos\mu}{8} \\ & + e^4 \,.\, \frac{\cos 4\mu - \cos 2\mu}{3} \\ & + e^5 \,.\, \frac{125\cos 5\mu - 135\cos 3\mu + 10\cos\mu}{384}.\end{aligned}$$

Using always the same symbols which stand for the elements of the disturbed planet, when accented, to represent the like elements of the disturbing planet, and introducing the new characters φ and φ' in order to shorten expressions, we shall have,

$$\upsilon - \alpha = \zeta + \varepsilon - \alpha + \sigma = \varphi + \sigma$$
$$\upsilon' - \alpha' = \zeta' + \varepsilon' - \alpha' + \sigma' = \varphi' + \sigma'.$$

If these values, as well the values of r and r', be substituted in that part of the disturbing function which is more easily dealt with, we shall obtain,

$$\frac{r\cos\omega}{r'^2} = \frac{a}{a'^2} \,.\, \left\{ \cos^2\frac{1}{2}\mathrm{I} \,.\, \cos(\varphi - \varphi' + \sigma - \sigma') \,.\, \frac{1-s}{(1-s')^2} \right.$$
$$\left. + \sin^2\frac{1}{2}\mathrm{I} \,.\, \cos(\varphi + \varphi' + \sigma + \sigma' - 2\nu) \,.\, \frac{1-s}{(1-s')^2} \right\};$$

or,

$$\frac{r\cos\omega}{r'^2} = \frac{a}{a'^2} \,.\, \left\{ \cos^2\frac{1}{2}\mathrm{I} \,.\, \cos(\varphi - \varphi') \,.\, \frac{(1-s)\cos(\sigma - \sigma')}{(1-s')^2} \right.$$
$$- \cos^2\frac{1}{2}\mathrm{I} \,.\, \sin(\varphi - \varphi') \,.\, \frac{(1-s)\sin(\sigma - \sigma')}{(1-s')^2}$$
$$+ \sin^2\frac{1}{2}\mathrm{I} \,.\, \cos(\varphi + \varphi' - 2\nu) \,.\, \frac{(1-s)\cos(\sigma + \sigma')}{(1-s')^2}$$
$$\left. - \sin^2\frac{1}{2}\mathrm{I} \,.\, \cos(\varphi + \varphi' - 2\nu) \,.\, \frac{(1-s)\sin(\sigma + \sigma')}{(1-s')^2} \,.\right\}$$

The inspecting of this formula is sufficient to show that the expansion of the part multiplied by $\sin^2\frac{1}{2}\mathrm{I}$ is deducible without calculation from the expansion of the other part. For, as σ' is a series of the sines, and s' a series of the cosines, of the multiples of μ', the former will change its sign with μ', while the latter will not vary: wherefore the expansion of the part multiplied by $\sin^2 \frac{1}{2} I$ will be obtained from the expansion of the other part, merely by making μ' negative, and writing $\varphi + \varphi' - 2\nu$ in place of $\varphi - \varphi'$ in all the arguments.

We have therefore only to expand the two first lines of the expression, for which purpose they may be thus written :

$$\cos(\varphi - \varphi') \cdot \left\{ (1-s)\cos\sigma \times \frac{\cos\sigma'}{(1-s')^2} + (1-s)\sin\sigma + \frac{(\sin\sigma')}{(1-s')^2} \right\}$$

$$+ \sin(\varphi - \varphi') \cdot \left\{ -(1-s)\sin\sigma \times \frac{\cos\sigma'}{(1-s')^2} + (1-s)\cos\sigma \times \frac{(\sin\sigma')}{(1-s')^2} \right\}$$

Neglecting quantities above the fifth order with respect to the excentricities, we have,

$$(1-s)\cos\sigma = 1 - s - \frac{\sigma^2}{2} + \frac{s\,\sigma^2}{2} + \frac{\sigma^4}{24} - \frac{s\,\sigma^4}{24},$$

$$(1-s)\sin\sigma = \sigma - s\,\sigma - \frac{\sigma^3}{6} + \frac{s\,\sigma^3}{6} + \frac{\sigma^5}{120};$$

And, if we assume,

$$(1-s)\cos\sigma = A^{(0)} + A^{(1)} e\cos\mu + A^{(2)} e^2 \cos 2\mu \ldots + A^{(5)} e^5 \cos 5\mu,$$

$$(1-s)\sin\sigma = B^{(1)} e\sin\mu + B^{(2)} e^2 \sin 2\mu \ldots . B^{(5)} e^5 \sin 5\mu;$$

we shall readily find,

$$A^{(0)} = 1 - \frac{e^2}{2} - \frac{e^4}{64} \qquad B^{(1)} = 2 - \frac{3}{8}e^2 + \frac{5}{96}e^4$$

$$A^{(1)} = -1 - \frac{3}{8}e^2 + \frac{5}{96}e^4 \qquad B^{(2)} = \frac{1}{4} - \frac{5}{12}e^2$$

$$A^{(2)} = \frac{1}{2} - \frac{e^2}{3} \qquad B^{(3)} = \frac{7}{24} - \frac{41}{96}e^2$$

$$A^{(3)} = \frac{3}{8} - \frac{13}{32}e^2 \qquad B^{(4)} = \frac{29}{64}$$

$$A^{(4)} = \frac{67}{129} \qquad B^{(5)} = \frac{77}{240}.$$

$$A^{(5)} \frac{17}{48}$$

And, by proceeding similarly,

$$\frac{\cos \sigma'}{(1-s')^2} = a^{(0)} + a^{(1)}.\, e' \cos \mu' + a^{(2)}.\, e'^2 \cos 2\mu' \ldots + a^{(5)}.\, e'^5 \cos 5\mu$$

$$\frac{\sin \sigma'}{(1-s')^2} = b^{(1)}.\, e' \sin \mu' + b^{(2)}.\, e'^2 \sin 2\mu' \ldots\ldots + b^{(5)}.\, e'^5 \sin 5\mu'$$

$$a^{(0)} = 1 - \frac{e'^2}{2} - \frac{e'^4}{24} \qquad b^{(1)}\, 2 - \frac{3}{2}\, e'^2 + \frac{5}{24}\, e'^4$$

$$a^{(1)} = 2 - \frac{3}{2}\, e'^2 + \frac{5}{24}\, e'^4 \qquad b^{(2)} = \frac{13}{4} - \frac{41}{12}\, e'^2$$

$$a^{(2)} = \frac{7}{2} - \frac{10}{3}\, e'^2 \qquad b^{(3)}\, \frac{31}{6} - \frac{161}{24}\, e'^2$$

$$a^{(3)} = \frac{11}{2} - \frac{53}{8}\, e'^2 \qquad b^{(4)} = \frac{761}{96}$$

$$a^{(4)} = \frac{1603}{192} \qquad b^{(5)} = \frac{2827}{240}.$$

$$a^{(5)} = \frac{149}{12}$$

We can now assign any term of the expansion we are considering. Let i and i' be any two positive numbers, zero included, of which the sum does not exceed 5: the part of the expansion multiplied by $e^i\, e'^{i'} \cos^2 \frac{1}{2} \mathrm{I}$ will be,

$$\cos(\varphi - \varphi') . \left\{ \mathrm{A}^{(i)}\, a^{(i')} \cos i\mu \cos i'\mu' + \mathrm{B}^{(i)}\, b^{(i')} \sin i\mu \sin i'\mu' \right\}$$

$$+ \sin(\varphi - \varphi') . \left\{ -\mathrm{B}^{(i)}\, a^{(i')} \sin i\mu \cos i'\mu' + b^{(i')}\, \mathrm{A}^{(i)} \sin i'\mu' \cos i\mu \right\}:$$

and, by reduction,

$$\left(\frac{\mathrm{A}^{(i)} a^{(i')} + \mathrm{B}^{(i)} b^{(i')}}{4} - \frac{\mathrm{B}^{(i)} a^{(i')} + b^{(i')} \mathrm{A}^{(i)}}{4} \right) . \cos(\varphi - \varphi' - i\mu + i'\mu')$$

$$+ \left(\frac{\mathrm{A}^{(i)} a^{(i')} + \mathrm{B}^{(i)} b^{(i')}}{4} + \frac{\mathrm{B}^{(i)} a^{(i')} + b^{(i')} \mathrm{A}^{(i)}}{4} \right) . \cos(\varphi - \varphi' + i\mu - i'\mu')$$

$$+ \left(\frac{\mathrm{A}^{(i)} a^{(i')} - \mathrm{B}^{(i)} b^{(i')}}{4} - \frac{\mathrm{B}^{(i)} a^{(i')} - b^{(i')} \mathrm{A}^{(i)}}{4} \right) . \cos(\varphi - \varphi' - i\mu - i'\mu')$$

$$+ \left(\frac{\mathrm{A}^{(i)} a^{(i')} - \mathrm{B}^{(i)} b^{(i')}}{4} + \frac{\mathrm{B}^{(i)} a^{(i')} - b^{(i')} \mathrm{A}^{(i)}}{4} \right) . \cos(\varphi - \varphi' + i\mu + i'\mu').$$

The other part of the expansion, that multiplied by $e^{(i)}\, e'^{(i')} \sin^2 \frac{1}{2} \mathrm{I}$, will be different in no respect from the part just computed except in the arguments of the cosines, which must be changed according to the rule already laid down; so that, instead of the four cosines above, these which follow must be substituted respectively,

$$\cos(\phi + \phi' - 2\nu - i\mu - i'\mu')$$
$$\cos(\phi + \phi' - 2\nu + i\mu + i'\mu')$$
$$\cos(\phi + \phi' - 2\nu - i\mu + i'\mu')$$
$$\cos(\phi + \phi' - 2\nu + i\mu - i'\mu').$$

Every part of the expansion being comprehended in the formula, it follows that all the arguments will contain the mean motions of the planets, excepting the particular case when $i = 1$, $i' = 1$: for, in this case, $\phi - \mu$ and $\phi' - \mu'$ are independent of the mean motions; and the first cosines of the two parts multiplied by $e\,e'$ will be,

$$\cos(\varpi - \varpi' - \alpha + \alpha')$$
$$\cos(\varpi + \varpi' - 2\nu - \alpha - \alpha').$$

These cosines have the same coefficient, viz.

$$\frac{\mathrm{A}^{(1)} a^{(1)} + \mathrm{B}^{(1)} b^{(1)}}{4} - \frac{\mathrm{B}^{(1)} a^{(1)} + \mathrm{A}^{(1)} b^{(1)}}{4};$$

And this will be found equal to zero, when the values of the symbols are substituted *. It thus appears, as far as the calculation has been carried, that there are no terms in the expansion of $\frac{r \cos \omega}{r'^2}$ except such as contain the mean motions and are periodical.

3. We next proceed to the expansion of $\frac{1}{\rho}$, which is the more difficult part of the problem. The value of $\cos\omega$, already investigated, may be thus written,

$$\cos\omega = \cos(\phi - \phi' + \sigma - \sigma')$$
$$- \sin^2 \tfrac{1}{2} \mathrm{I} . \left\{ \cos(\phi - \phi' + \sigma - \sigma') - \cos(\phi + \phi' + \sigma + \sigma' - 2\nu) \right\} :$$

* The coefficient is equal to zero, because $a^{(i)} = b^{(i)}$: and for the same reason the coefficient

$$\frac{\mathrm{A}^{(1)} a^{(1)} - \mathrm{B}^{(1)} b^{(1)}}{4} + \frac{\mathrm{B}^{(1)} a^{(1)} - b^{(1)} \mathrm{A}^{(1)}}{4}$$

is equal to zero.

wherefore,

$$D^2 = r^2 + r'^2 - 2rr' \cos(\varphi - \varphi' + \sigma - \sigma'),$$
$$q = \cos(\varphi - \varphi + \sigma - \sigma') - \cos(\varphi + \varphi' + \sigma + \sigma' - 2\nu),$$
$$\varrho^2 = r^2 + r'^2 - 2rr' \cos\omega = D^2 + 2\sin^2\tfrac{1}{2}I \, . \, rr' q:$$

consequently,

$$\frac{1}{\rho} = \frac{1}{D} - \sin^2\tfrac{1}{2}I \, . \, \frac{rr'}{D^3} q + \tfrac{3}{2}\sin^4\tfrac{1}{2}I \, . \, \frac{r^2 r'^2}{D^5} q^2;$$

and the three parts of $\frac{1}{\rho}$ must be expanded separately.

Expansion of $\frac{1}{D}$.

By substituting $a(1-s)$ and $a'(1-s')$ for r and r', we get

$$\frac{1}{D} = \frac{1}{\sqrt{a^2(1-s)^2 + a'^2(1-s')^2 - 2aa'(1-s)(1-s')\cos(\varphi - \phi' + \sigma - \sigma')}};$$

and it will be found that the following equation, in partial differentials, is true, viz.

$$\frac{1}{D} = \frac{d.\frac{1}{D}}{ds}(1-s) + \frac{d.\frac{1}{D}}{ds'}(1-s').$$

All the possible values of $\frac{1}{D}$ in this equation are comprehended in the formula,

$$\frac{1}{D} = G \times F;$$

provided F and G verify these equations,

$$G = \frac{dG}{ds}(1-s) + \frac{dG}{ds'}(1-s')$$

$$0 = \frac{dF}{ds}(1-s) + \frac{dF}{ds'}(1-s').$$

We may adopt for G any particular expression which verifies its proper equation; and then F will be determined by the nature of the quantity sought. Any function of $(1-s)$ and $(1-s')$ of -1 dimension will verify the equation for G; and, taking the most simple case, $G = \frac{1}{1-s}$, we shall have

$$\frac{1}{D} = \frac{1}{1-s} \times F.$$

Now let

$$D_0 = \sqrt{a^2 + a'^2 - 2\,a\,a' \cos(\phi - \phi' + \sigma - \sigma')};$$

then first, if we suppose $s = 0$, $s' = 0$, we get

$$\frac{1}{D} = \frac{1}{D_0};$$

and again, if we suppose $s = s'$, the result will be,

$$\frac{1}{D} = \frac{1}{1-s} \cdot \frac{1}{D_0}:$$

and thus we learn that

$$F = \frac{1}{D_0},$$

both when $s = 0$, $s' = 0$, and when $s = s'$. From its equation we know that F must be a function of no dimensions with respect to $(1 - s)$ and $(1 - s')$; and this condition, as well as what has been shown to take place on the two foregoing suppositions, will be fulfilled by making F a function of $\left(1 - \frac{1-s'}{1-s}\right) = \frac{s'-s}{1-s}$: wherefore the expression of $\frac{1}{D}$ will be as follows:

$$\frac{1}{D} = \frac{1}{1-s} \cdot f\left(\frac{s'-s}{1-s}\right),$$

f being the mark of a function which, in the present case, must be equal to $\frac{1}{D_0}$ when $\frac{s'-s}{1-s} = 0$.

Since s and s' are small quantities, the expression of $\frac{1}{D}$ may be expanded. By neglecting quantities above the fifth order with respect to s and s',

$$\frac{1}{D} = \frac{B^{(0)}}{1-s} + B^{(1)} \cdot \frac{s'-s}{(1-s)^2} + B^{(2)} \cdot \frac{(s'-s)^2}{(1-s)^3} \ldots\ldots B^{(5)} \cdot \frac{(s'-s)^5}{(1-s)^6}.$$

In this form it is obvious that all the coefficients may be readily found. For, by differentiating successively with respect to s', and supposing $s = 0$, $s' = 0$, in all the results, we obtain,

$$B^{(0)} = \frac{1}{D_0},\ B^{(1)} = \frac{d\frac{1}{D}}{d s'},\ B^{(2)} = \frac{1}{1.2} \cdot \frac{dd\frac{1}{D}}{d s'^2},\ \&c.$$

But the expression is complicated; and in order to reduce it to a more manageable form, every separate term may be resolved into a series of the powers and products of s and s': which being done, and the quantities of the same order classed together, we shall find,

$$\begin{aligned}\frac{1}{D}=&B^{(0)}+B^{(1)}.\,s' \quad +B^{(2)}s^{\prime 2} \quad +B^{(3)}s'^{3} \quad +\&c.;\\ &-\left(B^{(1)}-B^{(0)}\right).s-\left(B^{(2)}-B^{(1)}\right).2\,ss' \quad -\left(B^{(3)}-B^{(2)}\right).3\,s'^{2}s\\ &+\left(B^{(2)}-2B^{(1)}+B^{(0)}\right)s^{2}+\left(B^{(3)}-2B^{(2)}+B^{(1)}\right).3\,s's^{2}\\ &-\left(B^{(3)}-3B^{(2)}+3B^{(1)}-B^{(0)}\right).s^{3}\end{aligned}$$

Or, in the usual notation of finite differences,

$$\begin{aligned}\frac{1}{D}=&B^{(0)} \quad +B^{(1)}.s' \quad +B^{(2)}.s'^{2} \quad +B^{(3)}s'^{3} \quad +B^{(4)}..s'^{4} \quad +\&c.;\\ &-\Delta B^{(0)}.s-\Delta B^{(1)}.2ss'-\Delta B^{(2)}.3s'^{2}s \quad -\Delta B^{(3)}.4\,s'^{3}s\\ &+\Delta^{2}B^{0}.s^{2} \quad +\Delta^{2}B^{(1)}.3s's^{2} \quad +\Delta^{2}B^{(2)}.6\,s'^{2}s^{2}\\ &-\Delta^{3}B^{0}.s^{3} \quad -\Delta^{3}B^{(1)}.4\,s's^{3}\\ &+\Delta^{4}B^{(0)}.s^{4}.\end{aligned}$$

The value of $\frac{1}{D}$ seems now to be reduced to its most simple form as far as s and s' are concerned. If it be observed that, in every term, the exponent of the power of s' is the same with the numeral affix of B, and the exponent of s the same with the index of Δ, it will appear that the whole expansion is comprehended in the formula,

$$\Delta^{i}B^{(i')}.\,\beta s'^{i'}s^{i},$$

β being the coefficient of $s'^{i'}s^{i}$ in $(s'-s)^{i'+i}$, and i', i, representing the several positive numbers, zero included, of which the sum $(i'+i)$ does not exceed 5, when the expansion is limited to quantities of the fifth order.

4. The values of the quantities $B^{(0)}$, $B^{(1)}$, &c., are next to be investigated. If, for the sake of abridging, we put,

$$Q=a'^{2}(1-s')-a\,a'(1-s)\cos(\varphi-\varphi'+\sigma-\sigma'),$$

we shall get by repeated differentiations,

$$\frac{d\frac{1}{D}}{ds'} = \frac{Q}{D^3},$$

$$\frac{1}{1.2} \cdot \frac{d^2\frac{1}{D}}{ds'^2} = \frac{1}{2} \cdot \frac{dQ}{ds'} \cdot \frac{1}{D^3} + \frac{3}{2} \cdot \frac{Q^2}{D^5},$$

$$\frac{1}{1.2.3} \cdot \frac{d^3\frac{1}{D}}{ds'^3} = \frac{3}{2} \cdot \frac{dQ}{ds'} \cdot \frac{Q}{D^5} + \frac{5}{2} \cdot \frac{Q^3}{D^7},$$

$$\frac{1}{1.2.3.4} \cdot \frac{d^4\frac{1}{D}}{ds'^4} = \frac{3}{8} \cdot \left(\frac{dQ}{ds'}\right)^2 \cdot \frac{1}{D^5} + \frac{15}{4} \cdot \frac{dQ}{ds} \cdot \frac{Q^2}{D^7} + \frac{35}{8} \cdot \frac{Q^4}{D^9},$$

$$\frac{1}{1.2.3.4.5} \cdot \frac{d^5\frac{1}{D}}{ds'^5} = \frac{15}{8}\left(\frac{dQ}{ds}\right)^2 \cdot \frac{Q}{D^7} + \frac{35}{4} \cdot \frac{dQ}{ds} \cdot \frac{Q^3}{D^9} + \frac{63}{8} \cdot \frac{Q^5}{D^{11}}.$$

Now these expressions become equal to $B^{(1)}$, $B^{(2)}$, &c., when $s' = 0$, $s = 0$, that is, when

$$D = D_0,$$

$$Q = a'^2 - a\,a' \cos(\phi - \phi' + \sigma - \sigma') = \frac{D_0^2 + (a'^2 - a^2)}{2},$$

$$\frac{dQ}{ds'} = -\,a'^2.$$

Let us now put

$$f = a'^2 - a^2, \qquad h = \frac{a'^2 + a^2}{a'^2 - a^2};$$

then,

$$Q = \frac{D_0^2 + f}{2}, \qquad \frac{dQ}{ds} = -f \times \frac{1+h}{2}:$$

and, by substituting these values in the foregoing expressions, and reducing, we get

$$B^{(0)} = \frac{1}{D_0}$$

$$B^{(1)} = \frac{1}{2}\left(\frac{1}{D_0} + \frac{f}{D_0^3}\right)$$

$$B^{(2)} = \frac{3}{8}\left(\frac{1}{D_0} + \frac{f^2}{D_0^5}\right) + \frac{2-h}{4} \cdot \frac{f}{D_0^3},$$

$$B^{(3)} = \frac{5}{16}\left(\frac{1}{D_0} + \frac{f^3}{D_0^7}\right) + \frac{9-6h}{16}\left(\frac{f}{D^3} + \frac{f^2}{D^5}\right)$$

$$B^{(4)} = \frac{35}{128}\left(\frac{1}{D_0} + \frac{f^4}{D_0^9}\right) + \frac{20-15h}{32}\left(\frac{f}{D_0^3} + \frac{f^3}{D_0^7}\right) + \frac{6h^2 - 48h + 51}{64} \cdot \frac{f^2}{D_0^5}$$

$$B^{(5)} = \frac{63}{256}\left(\frac{1}{D_0} + \frac{f^5}{D_0^{11}}\right) + \frac{175-140h}{256}\left(\frac{f}{D_0^3} + \frac{f^4}{D_0^9}\right) + \frac{30h^2 - 150h + 135}{128} \cdot \left(\frac{f^2}{D_0^5} + \frac{f^3}{D_0^7}\right).$$

These quantities have been deduced from the differentials $\frac{d\frac{1}{D}}{ds'}$, $\frac{1}{1.2} \cdot \frac{dd\frac{1}{D}}{ds'^2}$, &c. If the differentials $\frac{d\frac{1}{D}}{ds}$, $\frac{1}{1.2} \cdot \frac{dd\frac{1}{D}}{ds^2}$, &c., be used, the same results will be found, with this difference, that f and h, retaining the same numerical values, will both change their signs.

If we write ψ for $(\varphi - \varphi' + \sigma - \sigma')$ and differentiate the formula,

$$\frac{d\frac{1}{D_0^n}}{d\psi} = -n \cdot \frac{aa' \sin\psi}{D_0^{n+2}},$$

the result will be,

$$\frac{dd\frac{1}{D_0^n}}{d\psi^2} = -n \cdot \frac{aa' \cos\psi}{D_0^{n+2}} + n \cdot n+2 \cdot \frac{a^2 a'^2 \sin^2\psi}{D_0^{n+4}}:$$

now,

$$aa' \cos\psi = \frac{a'^2 + a^2}{2} - \frac{D_0^2}{2},$$

$$a'^2 a^2 \sin^2\psi = -\frac{D_0^4}{4} + \frac{a'^2 + a^2}{2} \cdot D_0^2 - \frac{(a'^2 - a^2)^2}{4};$$

wherefore,

$$\frac{dd\frac{1}{D_0^n}}{d\psi^2} = -\frac{n^2}{4} \cdot \frac{1}{D_0^n} + \frac{n \cdot n+1}{2} \cdot \frac{a'^2 + a^2}{D_0^{n+2}} - \frac{n \cdot n+2}{4} \cdot \frac{(a'^2 - a^2)^2}{D_0^{n+4}}:$$

whence we deduce

$$\frac{f^2}{D_0^{n+4}} = -\frac{4}{n \cdot n+2} \cdot \frac{d\,d\frac{1}{D_0^n}}{d\,\psi^2} - \frac{n}{n+2} \cdot \frac{1}{D_0^n} + \frac{2(n+1)}{n+2} h \cdot \frac{f}{D_0^{n+2}}.$$

In this formula put n successively equal to 1, 3, 5, 7; then

$$\frac{f^2}{D_0^5} = -\frac{4}{3} \cdot \frac{d\,d\frac{1}{D_0}}{d\,\psi^2} - \frac{1}{3} \cdot \frac{1}{D_0} + \frac{4h}{3} \cdot \frac{f}{D_0^3},$$

$$\frac{f^3}{D_0^7} = -\frac{4}{15} \cdot \frac{d\,d\frac{f}{D_0^3}}{d\,\psi^2} - \frac{3}{5} \cdot \frac{f}{D_0^3} + \frac{8h}{5} \cdot \frac{f^2}{D_0^5},$$

$$\frac{f^4}{D_0^9} = -\frac{4}{35} \cdot \frac{d\,d\frac{f^2}{D_0^5}}{d\,\psi^2} - \frac{5}{7} \cdot \frac{f^2}{D_0^5} + \frac{12h}{7} \cdot \frac{f^3}{D_0^7},$$

$$\frac{f^5}{D_0^{11}} = -\frac{4}{63} \cdot \frac{d\,d\frac{f^3}{D_0^7}}{d\,\psi^2} - \frac{7}{9} \cdot \frac{f^3}{D_0^7} + \frac{16h}{9} \cdot \frac{f^4}{D_0^9}.$$

By means of the two first of these last formulas, we can exterminate $\frac{f^2}{D_0^5}$ and $\frac{f^3}{D_0^7}$ from the values of $B^{(2)}$ and $B^{(3)}$; which being done, we shall find

$$B^{(2)} = \left(\frac{1}{4} \cdot \frac{1}{D_0} - \frac{1}{2} \cdot \frac{d\,d\frac{1}{D_0}}{d\,\psi^2}\right) + \frac{2+h}{4} \cdot \frac{f}{D_0^3},$$

$$B^{(3)} = \left(\frac{3-h}{24} \cdot \frac{1}{D_0} - \frac{9+2h}{12} \cdot \frac{d\,d\frac{1}{D_0}}{d\,\psi^2}\right) + \left(\frac{4h^2+9h+9}{24} \cdot \frac{f}{D_0^3} - \frac{1}{12} \cdot \frac{d\,d\frac{f}{D_0^3}}{d\,\psi^2}\right).$$

And, in like manner, by means of all the four formulas, we may exterminate $\frac{f^2}{D_0^5}$, $\frac{f^3}{D_0^7}$, $\frac{f^4}{D_0^9}$ and $\frac{f^5}{D_0^{11}}$, from the expressions of $B^{(4)}$ and $B^{(5)}$; and thus are obtained these values,

$$B^{(4)} = -\frac{3h^2+8h-7}{96} \cdot \frac{1}{D_0} + \frac{3h^3+8h^2+8h+6}{24} \cdot \frac{f}{D_0^3}$$

$$- \frac{12h^2+32h+77}{96} \cdot \frac{d\,d\frac{1}{D_0}}{d\,\psi^2} - \frac{1}{6} \cdot \frac{d\,d\frac{f}{D_0^3}}{d\,\psi^2} - \frac{1}{32} \cdot \frac{d\,d\frac{f^2}{D_0^5}}{d\,\psi^2},$$

$$B^{(5)} = -\frac{24h^3+75h^2+92h-55}{960}\cdot\frac{1}{D_0}+\frac{96h^4+300h^3+341h^2+200h+159}{960}\cdot\frac{f}{D_0^3}$$
$$-\frac{96h^3+300h^2+368h+725}{960}\cdot\frac{dd\frac{1}{D_0^3}}{d\psi^2}-\frac{12h^2+221}{960}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2}$$
$$-\frac{25-4h}{320}\cdot\frac{dd\frac{f^2}{D_0^5}}{d\psi^2}-\frac{1}{64}\cdot\frac{dd\frac{f^3}{D_0^7}}{d\psi^2}.$$

If we take the second differentials relatively to ψ in the first two of the same four formulas, we shall obtain these values of $\frac{dd\frac{f^2}{D_0^5}}{d\psi^2}$ and $\frac{dd\frac{f^3}{D_0^7}}{d\psi^2}$, viz.

$$\frac{dd\frac{f^2}{D_0^5}}{d\psi^2}=-\frac{4}{3}\cdot\frac{d^4\frac{1}{D_0}}{d\psi^4}-\frac{1}{3}\cdot\frac{dd\frac{1}{D_0}}{d\psi^2}+\frac{4h}{3}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2},$$
$$\frac{dd\frac{f^3}{D_0^7}}{d\psi^2}=-\frac{4}{15}\cdot\frac{d^4\frac{f}{D_0^3}}{d\psi^4}-\frac{32h}{15}\cdot\frac{d^4\frac{1}{D_0}}{d\psi^4}-\frac{8h}{15}\cdot\frac{dd\frac{1}{D_0}}{d\psi^2}+\frac{32h^2-9}{15}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2}:$$

which being substituted, the expressions of $B^{(4)}$ and $B^{(5)}$ will be as follow:

$$B^{(4)}=-\frac{3h^2+8h-7}{96}\cdot\frac{1}{D_0}-\frac{3h^2+8h+19}{24}\cdot\frac{dd\frac{1}{D_0}}{d\psi^2}+\frac{1}{24}\cdot\frac{d^4\frac{1}{D_0}}{d\psi^4}$$
$$+\frac{3h^3+8h^2+8h+6}{24}\cdot\frac{f}{D_0^3}-\frac{h+4}{24}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2}.$$

$$B^{(5)}=\left\{-\frac{24h^3+75h^2+92h-55}{960}\cdot\frac{1}{D_0}-\frac{24h^3+75h^2+91h-175}{240}\cdot\frac{dd\frac{1}{D_0}}{d\psi^2}\right.$$
$$\left.+\frac{25+4h}{240}\cdot\frac{d^4\frac{1}{D_0}}{d\psi^4}\right\}$$
$$+\left\{\frac{96h^4+300h^3+341h^2+200h+159}{960}\cdot\frac{f}{D_0^3}-\frac{7h^2+25h+53}{240}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2}\right.$$
$$\left.+\frac{1}{240}\cdot\frac{d^4\frac{f}{D_0^3}}{d\psi^4}\right\}.$$

Thus the six quantities $B^{(0)}$, $B^{(1)}$, &c., are all expressed in terms of $\frac{1}{D_0}$

and $\frac{f}{D_0^3}$, and the second and fourth differentials of the same two quantities, the coefficients being of easy computation, and depending solely on the mean distances of the two planets.

5. The problem to be solved is, to expand $\frac{1}{D}$ in a series of terms arranged according to the cosines of the arc ψ, or $\varphi - \varphi' + \sigma - \sigma'$, and its successive multiples. Now as the quantities $\frac{1}{D_0}$ and $\frac{f}{D_0^3}$ and their differentials, are all susceptible of a like development, what is required will be accomplished merely by substituting in $B^{(0)}$, $B^{(1)}$, &c., the parts of $\frac{1}{D_0}$, $\frac{f}{D_0^3}$, and their differentials, multiplied by the cosine of the same multiple of ψ, instead of the quantities themselves. The coefficients of the cosines in the developments in question are functions of $\frac{a}{a'}$ or $\frac{a'}{a}$; and, for the sake of convergency, we must choose that one of the two fractions which is less than unit. Supposing that a' is greater than a, and that n represents any odd number, we have,

$$\frac{f^{\frac{n-1}{2}}}{D_0^{\,n}} = \frac{\left(1 - \frac{a^2}{a'^2}\right)^{\frac{n-1}{2}}}{a'} \times \frac{1}{\left\{1 + \frac{a^2}{a'^2} - 2\frac{a}{a'}\cos\psi\right\}^{\frac{n}{2}}}.$$

Wherefore, assuming as usual,

$$\frac{1}{\left\{1 + \frac{a^2}{a'^2} - 2\frac{a}{a'}\cos\psi\right\}^{\frac{n}{2}}} = \tfrac{1}{2}C_n^{\;0} + C_n^{(1)}\cos\psi + C_n^{(2)}\cos 2\psi \ldots + C_n^{(k)}\cos k\psi \ldots,$$

the part of $\frac{1}{D_0}$, and of its second and fourth differentials, which are multiplied by $\cos k\psi$, will be respectively,

$$\frac{\cos k\psi}{a'} \times C_1^{(k)},\ \frac{\cos k\psi}{a'} \times -k^2 C_1^{(k)},\ \frac{\cos k\psi}{a'} \times k^4 C_1^{(k)}:$$

and the like parts of $\frac{f}{D_0^3}$ and its differentials, will be,

$$\frac{1 - \frac{a^2}{a'^2}}{a'} \times C_3^{(k)} \cos k\psi,\ \frac{1 - \frac{a^2}{a'^2}}{a'} \times -k^2 C_3^{(k)} \cos k\psi,\ \frac{1 - \frac{a^2}{a'^2}}{a'} \times k^4 C_3^{(k)} \cos k\psi.$$

It remains then to substitute these values in $B^{(0)}$, $B^{(1)}$, &c., in place of $\frac{1}{D_0}$, $\frac{f}{D_0^3}$, and their differentials: but, in order to obtain the formulas most convenient for calculation, it will be requisite to express the coefficient of the development of $\frac{f}{D_0^3}$ by means of those of the development of $\frac{1}{D_0}$.

If we write α for $\frac{a}{a'}$, and put

$$V = \sqrt{1 + \alpha^2 - 2\,\alpha\cos\psi},$$

we have this formula,

$$\frac{d\,.\,\frac{1}{V}}{d\,\psi} = -\,\frac{\alpha\sin\psi}{V^3};$$

and if we substitute the expansions of $\frac{1}{V}$ and $\frac{1}{V^3}$, and equate the coefficients of $\sin k\,\psi$, the result will be

$$2\,k\,C_1^{(k)} = \alpha\,C_3^{(k-1)} - \alpha\,C_3^{(k-1)}.$$

We have likewise this identical expression,

$$\frac{1}{V} = \frac{V^2}{V^3} = \frac{1+\alpha^2}{V^3} - \frac{2\,\alpha\cos\psi}{V^3}:$$

and by substituting the expansions of $\frac{1}{V}$ and $\frac{1}{V^3}$, and equating the coefficients of $\cos k\,\psi$, we get

$$C_1^{(k)} = (1+\alpha^2)\,C_3^{(k)} - \alpha\,C_3^{(k-1)} - \alpha\,C_3^{(k+1)}.$$

By combining the two formulas, we deduce

$$2\,\alpha\,C_3^{(k+1)} = (1+\alpha^2)\,C_3^{(k)} - (2\,k+1)\,C_1^{(k)}$$

$$2\,\alpha\,C_3^{k-1} = (1+\alpha^2)\,C_3^{(k)} - (2\,k-1)\,C_1^{(k)}.$$

Change k for $k-1$ in the first of these formulas, and for $k+1$ in the second, and the two following formulas will be obtained,

$$2\,\alpha\,C_3^{(k)} = (1+\alpha^2)\,C_3^{k-1} - (2\,k-1)\,C_1^{(k-1)}$$

$$2\,\alpha\,C_3^{(k)} = (1+\alpha^2)\,C_3^{(k+1)} + (2\,k+1)\,C_1^{k+1}.$$

By combining the four last formulas in different ways, different values of $C_3^{(k)}$ will be found, which may be used at pleasure, viz.

$$(1-\alpha^2)\,C_3^{(k)} = (2\,k+1)\,.\left\{\frac{1+\alpha^2}{1-\alpha^2}\,C_1^{(k)} - \frac{2\,\alpha}{1-\alpha^2}\,.\,C_1^{(k+1)}\right\}$$

$$(1-\alpha^2)\,C_3^{(k)} = (2\,k-1)\,.\left\{\frac{2\,\alpha}{1-\alpha^2}\,C_1^{(k-1)} - \frac{1+\alpha^2}{1-\alpha^2}\,C_1^{(k)}\right\}$$

$$(1-\alpha^2)\,C_3^{(k)} = \frac{4\,k^2-1}{4\,k}\,.\,\frac{2\,\alpha}{1-\alpha^2}\,.\,\left(C_1^{k-1} - C^{k+1}\right).$$

In order to shorten expressions, and to bring the formulas to a form convenient for calculation, let

$$\alpha = \tan \tfrac{1}{2}\,\theta\,;$$

then,

$$\frac{1+\alpha^2}{1-\alpha^2} = h = \frac{1}{\cos\theta},\ \frac{2\,\alpha}{1-\alpha^2} = \frac{\sin\theta}{\cos\theta} = h\sin\theta:$$

further, assume the new symbol $a^{(k)}$, the value of which may be calculated by any one of these formulas, viz.

$$a^{(k)} = (2\,k+1)\,.\left(C_1^{k} - \sin\theta\,C_1^{(k+1)}\right),$$

$$a^{(k)} = (2\,k-1)\,.\left(\sin\theta\,C_1^{(k-1)} - C_1^{(k)}\right),$$

$$a^{(k)} = \frac{4\,k^2-1}{4\,k}\,.\,\sin\theta\left(C_1^{k-1} - C_1^{k+1}\right);$$

then,

$$(1-\alpha^2)\,C_3^{(k)} = h\,.\,a^{(k)}.$$

Using this value, the parts of $\frac{f}{D_0^{\,3}}$ and its differentials, multiplied by $\cos k\,\psi$ will be

$$\frac{\cos k\,\psi}{a'}\times h\,a^{(k)},\ \frac{\cos k\,\psi}{a'}\times -\,h\,k^2\,a^{(k)},\ \frac{\cos k\,\psi}{a'}\times h\,k^4\,a^{k}.$$

These, as well as the like parts of $\frac{1}{D_0}$ and its differentials, must be substituted in $B^{(0)}$, $B^{(1)}$, &c. As there is no longer occasion to refer to any development but one, C^k may be written for $C_1^{(k)}$. In substituting we omit the common factor $\frac{\cos k\,\psi}{a'}$, and put

$$b_0^{(k)},\ b_1^{(k)},\ b_2^{(k)}\,\ldots\ldots\,b_5^{(k)}$$

for the expressions into which

$$B^{(0)}, B^{(1)}, B^{(2)} \ldots\ldots B^{(5)}$$

are changed, so that,

$$b_0^{(k)} = C^{(k)}$$

$$b_1^{(k)} = \frac{1}{2} C^{(k)} + \frac{h}{2} . a^{(k)},$$

$$b_2^{(k)} = \left(\frac{1}{4} + \frac{1}{2} k^2\right) C^{(k)} + \frac{h^2 + 2h}{4} . a^{(k)},$$

$$b_3^{(k)} = \left(\frac{3-h}{24} + \frac{9+2h}{12} . k^2\right) . C^{(k)} + \left(\frac{4h^3 + 9h^2 + 9h}{24} + \frac{h}{12} k^2\right) . a^{(k)},$$

$$b_4^{(k)} = \left(-\frac{3h^2 + 8h - 7}{96} + \frac{3h^2 + 8h + 19}{24} . k^2 + \frac{k^4}{24}\right) . C^{(k)}$$

$$+ \left(\frac{3h^4 + 8h^3 + 8h^2 + 6h}{24} + \frac{h^2 + 4h}{24} . k^2\right) . a^{(k)},$$

$$b_5^{(k)} = \left(-\frac{24h^3 + 75h^2 + 92h - 55}{960} + \frac{24h^3 + 75h^2 + 91h + 175}{240} . k^2\right.$$

$$\left. + \frac{25 + 4h}{240} . k^4\right) . C^{(k)}$$

$$+ \left(\frac{96h^5 + 300h^4 + 341h^3 + 200h^2 + 159h}{960} + \frac{7h^3 + 25h^2 + 53h}{240} k^2 + \frac{h k^4}{240}\right) . a^{(k)}.$$

And, these symbols being established, the part of $\frac{1}{D}$ we are in search of, that is, the part multiplied by $\cos k(\phi - \phi' + \sigma - \sigma')$, will be thus expressed,

$$\frac{\cos k(\phi - \phi' + \sigma - \sigma')}{a'} \times$$

$$\begin{aligned} \Big\{ & b_0^{(k)} + b_1^{(k)} . s^1 + b_2^{(k)} . s'^2 + b_3^{(k)} . s'^3 + b_4^{(k)} . s'^4 + b_5^{(k)} . s'^5 \\ & - \Delta b_0^{k} . s - \Delta b_1^{(k)} . 2 s^1 s - \Delta b_2^{(k)} . 3 s'^2 s - \Delta b_3^{(k)} . 4 s'^3 s - \Delta b_4^{(k)} . 5 s'^4 s \\ & + \Delta^2 b_0^{(k)} . s^2 + \Delta^2 b_1^{(k)} . 3 s^1 s^2 + \Delta^2 b_2^{(k)} . 6 s'^2 s^2 + \Delta^2 b_3^{(k)} . 10 s'^3 s^2 \\ & - \Delta^3 b_0^{(k)} . s^3 - \Delta^3 b_1^{(k)} . 4 s^1 s^3 - \Delta^3 b_2^{(k)} . 10 s'^2 s^3 \\ & + \Delta^4 b_0^{(k)} . s^4 + \Delta^4 b_1^{(k)} . 5 s^1 s^4 \\ & - \Delta^5 b_0^{(k)} . s^5. \end{aligned}$$

By the foregoing analysis the problem is reduced to a series of terms comprehended in the formula,

$$\pm \Delta^{i}\, b_{i'}^{(k)} \times \beta\, s'^{i'}\, s^{i} \cos k\,(\varphi - \varphi' + \sigma - \sigma')$$

i' and i representing all positive numbers, zero included, of which the sum $i' + i$ does not exceed 5; β standing for the coefficient of $s'^{i'} s^{i}$ in $(s' + s)^{i'+i}$; and the upper or lower sign taking place according as the index of Δ is even or odd. Although all the coefficients are supposed to be deduced from six of them, for which alone direct expressions are given, it is nevertheless obvious that an independent formula may be found by which any proposed coefficient may be separately computed: namely, by substituting the values of $b_0^{(k)}$, $b_1^{(k)}$, &c., in the expressions of the several finite differences.

Although there can be no difficulty in applying the formulas, it may not be improper, for the sake of illustration, to take an example. We shall choose the instance of

Venus and the Earth.

In this case a' will represent the mean distance of the Earth, and a that of Venus: wherefore,

$$\frac{a}{a'} = \alpha = 0{\cdot}7323332 = \tan \tfrac{1}{2}\theta, \quad \theta = 71^\circ\, 45'\, 32'',$$

$$h = \frac{1}{\cos\theta} = 3{\cdot}1947202, \quad \log.\sin\theta = 9{\cdot}9776083.$$

The method likewise requires that there be previously calculated a sufficient number of the first coefficients in the expansion,

$$\frac{1}{\sqrt{1 + \alpha^2 - 2\,\alpha\cos\psi}} = \tfrac{1}{2}\,\mathrm{C}^{(0)} + \mathrm{C}^{(1)}\cos\psi + \mathrm{C}^{(2)}\cos 2\,\psi \ldots\ldots \mathrm{C}^{(k)}\cos k\,\psi \ldots\ldots,$$

which is accomplished by rules amply detailed in all the treatises on Physical Astronomy. We next deduce the value of $\mathrm{a}^{(k)}$, viz.

$$\mathrm{a}^{(k)} = (2\,k + 1)\,.\left(\mathrm{C}^{(k)} - \sin\theta\,\mathrm{C}^{(k+1)}\right):$$

and, with the numbers thus found, we have generally for any proposed value of k,

$$b_0^{(k)} = C^{(k)},$$

$$b_1^{(k)} = \tfrac{1}{2} C^{(k)} + 1{\cdot}59736 \,.\, a^{(k)}$$

$$b_2^{(k)} = \left(\frac{1}{4} + \frac{k^2}{2}\right) . C^{(k)} + 4{\cdot}148919 \,.\, a^{(k)}$$

$$b_3^{(k)} = \{- 0{\cdot}008113 + 1{\cdot}282453 \,.\, k^2\} . C^{(k)}$$

$$+ \{10{\cdot}459698 + 0{\cdot}266227\, k^2\} \,.\, a^{(k)}$$

$$b_4^{(k)} = \left\{- 0{\cdot}512255 + 3{\cdot}132352\, k^2 + \frac{k^4}{24}\right\} . C^{(k)}$$

$$+ \{28{\cdot}09035 + 0{\cdot}957713\, k^2\} \,.\, a^{(k)}$$

$$b_5^{(k)} = \{-1{\cdot}861383 + 8{\cdot}390554\, k^2 + 0{\cdot}157412\, k^4\} . C^{(k)}$$

$$+ \{80{\cdot}06813 + 2{\cdot}719660\, k^2 + {\cdot}0133113\, k^4\} \,.\, a^{(k)}$$

Nothing can be more easy than the computation of these six quantities for any assigned value of k. The other quantities, being the several orders of the finite differences of the first six, will be known when these have been computed: but it will be convenient and will often save much calculation to have an independent formula for every coefficient separately.

$$- \Delta b_0^{(k)} = \tfrac{1}{2} C^{(k)} - 1{\cdot}59736 \,.\, a^{(k)}$$

$$- \Delta b_1^{(k)} = \left(\frac{1}{4} - \frac{k^2}{2}\right) C^{(k)} - 2{\cdot}551559 \,.\, a^{(k)}$$

$$- \Delta b_2^{(k)} = \{0{\cdot}258113 - 0{\cdot}782453\, k^2\} . C^{(k)}$$

$$- \{6{\cdot}31078 + 0{\cdot}266227\, k^2\} \,.\, a^{(k)}$$

$$- \Delta b_3^{(k)} = \left\{0{\cdot}504142 - 1{\cdot}849899\, k^2 - \frac{k^4}{24}\right\} C^{(k)}$$

$$- \{17{\cdot}63065 + 0{\cdot}691486\, k^2\} \,.\, a^{(k)}$$

$$-\Delta b_4^{(k)} = \{1{\cdot}34913 - 5{\cdot}258202\,k^2 - 0{\cdot}115745\,k^4\}\,C^{(k)}$$

$$- \{51{\cdot}97778 + 1{\cdot}761947\,k^2 + {\cdot}0133113\,k^4\}\,.\,a^{(k)}$$

$$\Delta^2 b_0^{(k)} = \left(\frac{1}{4} + \frac{k^2}{2}\right) C^{(k)} + 0{\cdot}954199\,.\,a^{(k)}$$

$$\Delta^2 b_1^{(k)} = \{-{\cdot}008113 + {\cdot}282453\,k^2\}\,.\,C^{(k)}$$

$$+ \{3{\cdot}75922 + {\cdot}266227\,k^2\}\,.\,a^{(k)}$$

$$\Delta^2 b_2^{(k)} = \{-{\cdot}24603 + 1{\cdot}067446\,k^2 + {\cdot}041667\,k^4\}\,.\,C^{(k)}$$

$$+ \{11{\cdot}31987 + 0{\cdot}425259\,k^2\}\,.\,a^{(k)}$$

$$\Delta^2 b_3^{(k)} = \{-{\cdot}84499 + 3{\cdot}408303\,k^2 + {\cdot}074078\,k^4\}\,.\,C^{(k)}$$

$$+ \{34{\cdot}34714 + 1{\cdot}070461\,k^2 + {\cdot}0133113\,k^4\}\,a^{(k)}$$

$$-\Delta^3 b_0^{(k)} = \{0{\cdot}25811 + {\cdot}217547\,k^2\}\,.\,C^{(k)}$$

$$- \{2{\cdot}80502 + {\cdot}266227\,k^2\}\,.\,a^{(k)}$$

$$-\Delta^3 b_1^{(k)} = \{{\cdot}23792 - {\cdot}784993\,k^2 - {\cdot}041667\,k^4\}\,.\,C^{(k)}$$

$$- \{7{\cdot}56065 + {\cdot}159032\,k^2\}\,.\,a^{(k)}$$

$$-\Delta^3 b_2^{(k)} = \{{\cdot}59896 - 2{\cdot}340857\,k^2 - {\cdot}032411\,k^4\}\,.\,C^{(k)}$$

$$- \{23{\cdot}02727 + {\cdot}645158\,k^2 + {\cdot}013113\,k^4\}\,.\,a^{(k)}$$

$$\Delta^4 b_0^{(k)} = \{{\cdot}020197 + 1{\cdot}002540\,k^2 + {\cdot}041667\,k^4\}\,C^{(k)}$$

$$+ \{4{\cdot}75563 - {\cdot}107195\,k^2\}\,.\,a^{(k)}$$

$$\Delta^4 b_1^{(k)} = \{-{\cdot}36104 + 1{\cdot}555864\,k^2 - {\cdot}009256\,k^4\}\,.\,C^{(k)}$$

$$+ \{15{\cdot}46662 + {\cdot}486126\,k^2 + {\cdot}0133113\,k^4\}\,.\,a^{(k)}$$

$$-\Delta^5 b_0^{(k)} = \{{\cdot}38124 - {\cdot}553324\,k^2 + {\cdot}050923\,k^4\}\,C^{(k)}$$

$$- \{10{\cdot}71099 + {\cdot}593321\,k^2 + {\cdot}0133113\,k^4\}\,.\,a^{(k)}.$$

According to these formulas the calculation of any of the coefficients is an easy arithmetical process. In order to be able to compute, in any term of the expansion of $\frac{1}{D}$, the part of any proposed order with respect to e and e', nothing more is wanting than a method for reducing

$$\beta\, s_{\prime}^{i'}\, s^{i} \cos (\varphi - \varphi' + \sigma - \sigma')$$

to a series of simple cosines. On this point some observations will be offered below: but, without proceeding further, there is no difficulty in this respect, when we confine our views to quantities not passing the second or third order, which comprehends all the perturbations useful in astronomy, except the inequalities of very long periods.

If we combine the planets two and two, as is done in the sixth chapter of the sixth book of the Mécanique Céleste, and for every two planets construct a formula such as is exhibited above for Venus and the Earth, limiting the extent of the calculation according to the nature of the case, the theory of the planetary disturbances would be rendered more accessible, and would be freed from the tedious and disgusting labour which has rendered astronomers averse from cultivating this branch of their science.

6. The expression,

$$s_{\prime}^{i'}\, s^{i} \cos k (\varphi - \varphi' + \sigma - \sigma')$$

may be reduced to this form,

$$\cos k (\varphi - \varphi') \cdot \left\{ (s^{i} \cos k\sigma)(s_{\prime}^{i'} \cos k\sigma') + (s^{i} \sin k\sigma)(s_{\prime}^{i'} \sin k\sigma') \right\}$$

$$- \sin k (\varphi - \varphi') \cdot \left\{ (s^{i} \sin k\sigma)(s_{\prime}^{i'} \cos k\sigma') - (s^{i} \cos k\sigma)(s_{\prime}^{i'} \sin k\sigma') \right\}.$$

Here the quantities within the brackets are serieses, of which each contains the mean anomaly of only one planet, and comes under one or other of eleven different forms, namely, the six,

$$\cos k\sigma,\ s \cos k\sigma,\ s^2 \cos k\sigma, \ldots\ldots s^5 \cos k\sigma,$$

and the five,

$$\sin k\sigma,\ s \sin k\sigma,\ s^2 \sin k\sigma, \ldots\ldots s^4 \sin k\sigma,$$

the quantity $s^5 \sin k\sigma$, which is of the sixth order, being omitted. None of these serieses, when carried to quantities of the fifth order inclusively, and arranged according to the sines and cosines of the multiples of the mean anomaly, contain more than six terms, and they thus afford a ready way of calculating the part of

$$s'^{i'} s^{i} \cos k(\varphi - \varphi' + \sigma - \sigma')$$

of any proposed order with respect to e and e'.

These serieses are as follow :

$$\begin{aligned}
\text{Cos}\, k\sigma = {} & \left(1 - k^2 e^2 - \frac{9}{64} k^2 e^4 + \frac{k^4 e^4}{4}\right) \\
& - \left(\frac{5}{4} k^2 e^2 + \frac{1}{16} k^2 e^4 - \frac{5}{12} k^4 e^4\right) . e \cos \mu \\
& + \left(k^2 - \frac{4}{3} k^2 e^2 - \frac{k^4 e^2}{3}\right) . e^2 \cos 2\mu \\
& + \left(\frac{5}{4} k^2 - \frac{27}{16} k^4 e^2 - \frac{5}{8} k^4 e^2\right) . e^3 \cos 3\mu \\
& + \left(\frac{k^4}{12} + \frac{283}{192} k^2\right) . e^4 \cos 4\mu \\
& + \left(\frac{5k^4}{24} + \frac{7k^2}{4}\right) . e^5 \cos 5\mu .
\end{aligned}$$

$$\begin{aligned}
\text{Sin}\, k\sigma = {} & \left(2k - \frac{k e^2}{4} + \frac{5 k e^4}{96} - k^3 e^2 + \frac{13 k^3 e^4}{96} + \frac{k^5 e^5}{6}\right) . e \sin \mu \\
& + \left(\frac{5k}{4} - \frac{11 k e^2}{24} - \frac{5 k^3 e^2}{4}\right) . e^2 \sin 2\mu \\
& + \left(\frac{13k}{24} - \frac{43}{64} k e^2 - \frac{k^3}{3} - \frac{307 k^3 e^2}{192} - \frac{k^4 e^4}{12}\right) . e^3 \sin 3\mu \\
& + \left(\frac{103}{96} k + \frac{5}{8} k^3\right) . e^4 \sin 4\mu \\
& + \left(\frac{1097}{960} k + \frac{179}{192} k^3 + \frac{k^5}{60}\right) . e^5 \sin 5\mu .
\end{aligned}$$

$$\begin{aligned}
s \cos k\sigma = {} & -\frac{e^2}{2} + \frac{k^2 e^4}{8} \\
& + \left(1 - \frac{3e^2}{8} + \frac{5e^4}{192} - \frac{k^2 e^2}{2} + \frac{37 k^2 e^4}{96} + \frac{k^4 e^4}{12}\right) . e \cos \mu \\
& + \left(\frac{1}{2} - \frac{e^2}{3} - k^2 e^2\right) . e^2 \cos 2\mu
\end{aligned}$$

$$+\left(\frac{3}{8}-\frac{45\,e^2}{128}+\frac{k^2}{2}-\frac{183\,.\,k^2\,e^2}{128}-\frac{k^4\,e^2}{8}\right).\,e^3\cos 3\,\mu$$

$$+\left(\frac{1}{3}+\frac{7\,k^2}{8}\right).\,e^4\cos 4\,\mu$$

$$+\left(\frac{125}{384}+\frac{475}{384}\,k^2+\frac{k^4}{24}\right).\,e^5\cos 5\,\mu$$

$$s\sin k\,\sigma=-\,k\left(\frac{7}{8}\,e^2-\frac{3}{32}\,e^4-\frac{5\,k^2\,e^2}{24}\right).\,e\sin\mu$$

$$+\,k\left(1-\frac{23}{24}\,e^2-\frac{k^2\,e^2}{3}\right).\,e^2\sin 2\,\mu$$

$$+\,k\left(\frac{9}{8}-\frac{115\,.\,e^2}{96}-\frac{35\,k^2\,e^2}{48}\right).\,e^3\sin 3\,\mu$$

$$+\left(\frac{59}{48}\,k+\frac{k^3}{6}\right).\,e^4\sin 4\,\mu$$

$$+\left(\frac{11}{8}\,k+\frac{19}{48}\,k^3\right).\,e^5\sin 5\,\mu$$

$$s^2\cos k\,\sigma=\frac{e^2}{2}-\frac{k^2\,e^4}{4}$$

$$-\left(\frac{e^2}{2}-\frac{e^4}{24}+\frac{k^2\,e^4}{8}\right).\,e\cos\mu$$

$$+\left(\frac{1}{2}-\frac{e^2}{2}\right).\,e^2\cos 2\,\mu$$

$$+\left(\frac{1}{2}-\frac{9\,e^2}{16}-\frac{7\,k^2\,e^2}{16}\right).\,e^3\cos 3\,\mu$$

$$+\left(\frac{1}{2}+\frac{k^2}{4}\right).\,e^4\cos 4\,\mu$$

$$+\left(\frac{25}{48}+\frac{9\,k^2}{16}\right).\,e^5\cos 5\,\mu$$

$$s^2\sin k\,\sigma=\left(\frac{k\,e^2}{2}+\frac{k\,e^4}{12}-\frac{k^3 e^4}{12}\right).\,e\sin\mu$$

$$-\,\frac{3\,k\,e^2}{8}.\,e^2\sin 2\,\mu$$

$$+\left(\frac{k}{2}-\frac{5\,k\,e^2}{6}-\frac{k^3\,e^2}{12}\right).\,e^3\sin 3\,\mu$$

$$+ \frac{13\,k}{16} \,.\, e^4 \sin 4\,\mu$$

$$+ \left(\frac{13\,k}{12} + \frac{k^3}{12}\right) .\, e^5 \sin 5\,\mu$$

$$s^3 \cos k\,\sigma = -\frac{3}{8}\,e^4$$

$$+ \left(\frac{3}{4}\,e^2 - \frac{3\,e^4}{16} + k^2\,e^4\right) .\, e \cos \mu$$

$$+ \left(\frac{1}{4} - \frac{9\,e^2}{32} + \frac{k^2\,e^2}{8}\right) .\, e^3 \cos 3\,\mu$$

$$+ \frac{3}{8} \,.\, e^4 \cos 4\,\mu$$

$$+ \left(\frac{15}{32} + \frac{k^2}{8}\right) .\, e^5 \cos 5\,\mu$$

$$s^3 \sin k\,\sigma = -\frac{7\,k\,e^4}{16} \,.\, e \sin \mu + \frac{k\,e^2}{2} \,.\, e^2 \sin 2\,\mu + \frac{3\,k\,e^2}{32} \,.\, e^3 \sin 3\,\mu$$

$$+ \frac{k}{2} \,.\, e^4 \sin 4\,\mu + \frac{17\,k}{32} \,.\, e^5 \sin 5\,\mu$$

$$s^4 \cos k\,\sigma = \frac{3}{8} e^4 - \frac{e^4}{2} \,.\, e \cos \mu + \frac{e^2}{2} \,.\, e^2 \cos 2\,\mu + \frac{e^2}{4} \,.\, e^3 \cos 3\,\mu$$

$$+ \frac{1}{8} \,.\, e^4 \cos 4\,\mu + \frac{1}{4} \,.\, e^5 \cos 5\,\mu$$

$$s^4 \sin k\,\sigma = \frac{k\,e^4}{4} \,.\, e \sin \mu + \frac{3\,k\,e^2}{8} \,.\, e^3 \sin 3\,\mu + \frac{1}{8} \,.\, e^5 \sin 5\,\mu$$

$$s^5 \cos k\,\sigma = \frac{5\,e^4}{8} \,.\, e \cos \mu + \frac{5\,e^2}{16} \,.\, e^3 \cos 3\,\mu + \frac{1}{16}\, e^5 \cos 5\,\mu$$

The use of these serieses is obvious. If we wish to determine all the arguments of the order $e^{x} e'^{x'}$ in the term multiplied by $\cos k\,(\varphi - \varphi' + \sigma - \sigma')$, of the development of $\frac{1}{D}$, x and x' representing two positive integer numbers, zero included, of which the sum does not exceed 5; we first set aside all the parts of the expression in which i is greater than x, or i' greater than x', for these contain no quantities of the order $\overset{x}{e}\ \overset{x'}{e'}$. Let

$$\overset{i}{\Delta}\ b^{(k)}_{i'} \times \beta \overset{i}{s}\ \overset{i'}{s'} \cos k\,(\varphi - \varphi' + \sigma - \sigma')$$

be one of the remaining parts, in which i is not greater than x, nor i' greater than x'; in the serieses $\overset{i}{s} \cos k\,\sigma$ and $\overset{i}{s} \sin k\,\sigma$, take the terms $\mathrm{A}\,\overset{x}{e} \cos x\,\mu$ and $\mathrm{B}\,\overset{x}{e} \sin x\,\mu$; and, in $\overset{i'}{s'} \cos k\,\sigma'$ and $\overset{i'}{s'} \sin k\,\sigma'$, take the terms $\mathrm{A}'\,\overset{x'}{e'} \cos x'\,\mu'$ and $\mathrm{B}'\,\overset{x'}{e'} \sin x'\,\mu'$; then all the arguments of the order $\overset{x}{e}\ \overset{x'}{e'}$ will be contained in the formula

$$\overset{i}{\Delta}\ b^{(k)}_{i'} \times \beta \overset{x}{e}\ \overset{x'}{e'} \times$$

$$\cos k\,(\varphi - \varphi') \,.\, \{\mathrm{A}\,\mathrm{A}' \cos x\,\mu \cos x'\,\mu' + \mathrm{B}\,\mathrm{B}' \sin x\,\mu \sin x'\,\mu'\}$$
$$- \sin k\,(\varphi - \varphi') \,.\, \{\mathrm{A}'\,\mathrm{B} \sin x\,\mu \cos x'\,\mu' - \mathrm{A}\,\mathrm{B}' \cos x\,\mu \sin x'\,\mu'\}.$$

And thus are computed all the quantities of the order $\overset{x}{e}\ \overset{x'}{e'}$ in any term of the development of $\frac{1}{\mathrm{D}}$. The procedure is exactly the same with that followed in the expansion of the first part of the disturbing function in § 2. All the arguments of the order $\overset{x}{e}\ \overset{x'}{e'}$ are only four, comprehended in the formula

$$k\,(\varphi - \varphi') \pm x\,\mu \pm x'\,\mu',$$

which represents all the different ways of combining $x\,\mu$ and $x'\,\mu'$, by addition and subtraction, with $k\,(\varphi - \varphi')$. Hence it is easy to ascertain the orders of the same argument, or the relative magnitude of its coefficients, as it recurs in different terms of the development of $\frac{1}{\mathrm{D}}$.

7. We have next to consider the parts of the disturbing function which depend on the inclination of the orbits.

Development of $\sin^2 \frac{1}{2}\mathrm{I} \,.\, \frac{r\,r'}{\mathrm{D}^3} \,.\, q$.

Omitting quantities above the third order with respect to s and s' in the expression of $\frac{1}{\mathrm{D}}$ found in § 3, we have

$$\frac{1}{D} = B^{(0)} + B^{(1)} s' + B^{(2)} s'^2 \quad + B^{(3)} s'^3$$
$$- \Delta B^{(0)} s - \Delta B^{(1)} 2 s s' - \Delta B^{(2)} 3 s s'^2$$
$$+ \Delta^2 B^{(0)} s^2 \quad + \Delta^2 B^{(1)} 3 s^2 s'$$
$$- \Delta^3 B^{(0)} s^3.$$

As the values of $B^{(0)}$, $B^{(1)}$, &c., contain $\frac{1}{D_0}$ and its powers, they may be regarded as functions of cos ψ: taking the differentials with regard to this variable, we get,

$$\frac{d B^{(0)}}{d(\cos\psi)} = \frac{a a'}{D_0^3} = \frac{a a'}{f} \times \frac{f}{D_0^3} = \frac{a a'}{f} \times B'^{(0)},$$

$$\frac{d B^{(1)}}{d(\cos\psi)} = \frac{a a'}{f}\left(\frac{1}{2} \cdot \frac{f}{D_0^3} + \frac{3}{2} \cdot \frac{f^2}{D_0^5}\right) = \frac{a a'}{f} \times B'^{(1)},$$

$$\frac{d B^{(2)}}{d(\cos\psi)} = \frac{a a'}{f}\left(\frac{3}{8} \frac{f}{D_0^3} + \frac{6 - 3h}{4} \cdot \frac{f^2}{D_0^5} + \frac{15}{8} \cdot \frac{f^3}{D_0^7}\right) = \frac{a a'}{f} \times B'^{(2)},$$

$$\frac{d B^{(3)}}{d(\cos\psi)} = \frac{a a'}{f} \times \left(\frac{5}{16} \cdot \frac{f}{D_0^3} + \frac{27 - 18h}{16} \cdot \frac{f^2}{D_0^5} + \frac{45 - 30h}{16} \cdot \frac{f^3}{D_0^7} + \frac{35}{16} \cdot \frac{f^4}{D_0^9}\right)$$
$$= \frac{a a'}{f} \times B'^{(3)}.$$

If we now differentiate the above formula for $\frac{1}{D}$, and observe that

$$\frac{a a'}{f} = \frac{a a'}{a'^2 - a^2} = \frac{1}{2}\sqrt{h^2 - 1} = \frac{\tan\theta}{2},$$

the result, when the new symbols are introduced, will be as follows:

$$\frac{r r'}{D^3} = \frac{\tan.\theta}{2} \times \Big\{ B'^{(0)} + B'^{(1)} s^1 \quad + B'^{(2)} s'^2 \quad + B'^{(3)} s'^3$$
$$- \Delta B'^{(0)} s - \Delta B'^{(1)} . 2 s s^1 - \Delta B'^{(2)} 8 s s'^2$$
$$+ \Delta^2 B'^{(0)} s^2 \quad + \Delta^2 B'^{(1)} 3 s^2 s^1$$
$$- \Delta^3 B'^{(0)} s^3.$$

The values of $B'^{(0)}, B'^{(1)}$, &c., may be reduced to expressions that contain only $\frac{1}{D_0}$ and $\frac{f}{D_0^3}$ and their second and fourth differentials, by the same process that was applied to $B^{(0)}$, $B^{(1)}$, &c., in § 4: and in this manner we obtain,

$$B'^{(0)} = \frac{f}{D_0^3}$$

$$B'^{(1)} = -\frac{1}{2}\cdot\frac{1}{D_0} - 2\,\frac{dd\frac{1}{D_0}}{d\psi^2} + \frac{4h+1}{2}\cdot\frac{f}{D_0^3},$$

$$B'^{(2)} = -\frac{3h+2}{4}\cdot\frac{1}{D_0} - (3h+2)\,.\,\frac{dd\frac{1}{D_0}}{d\psi^2}$$

$$+\left(3h^2+2h-\frac{3}{4}\right)\cdot\frac{f}{D_0^3} - \frac{1}{2}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2}$$

$$B'^{(3)} = -\frac{24h^2+27h+1}{24}\cdot\frac{1}{D_0} - \frac{48h^2+54h+1}{12}\cdot\frac{dd\frac{1}{D_0}}{d\psi^2} + \frac{1}{3}\cdot\frac{d^4\frac{1}{D_0}}{d\psi^4}$$

$$+\frac{96h^3+108h^2-23h-33}{24}\cdot\frac{f}{D_0^3} - \frac{10h+9}{12}\cdot\frac{dd\frac{f}{D_0^3}}{d\psi^2}.$$

The term multiplied by $\cos k\psi$ in the expansion of $\frac{rr'}{D^3}$, will now be obtained by substituting in $B'^{(0)}$, $B'^{(1)}$, &c., the parts of $\frac{1}{D_0}$, $\frac{f}{D_0^3}$, and their differentials, multiplied by the same cosine. Let

$$\frac{\cos k\psi}{a'}\times b'^{(k)}_0,\ \frac{\cos k\psi}{a'}\times b'^{(k)}_1,\ \frac{\cos k\psi}{a'}\times b'^{(k)}_2,\ \frac{\cos k\psi}{a'}\times b'^{(k)}_3,$$

denote the expressions into which

$$B'^{(0)}, B'^{(1)}, B'^{(2)}, B'^{(3)},$$

are changed by the substitutions mentioned; the quantity sought, or the part of the expansion of $\sin^2\frac{1}{2}I\times\frac{rr'}{D^3}\times q$ multiplied by $\cos k\psi$, will be thus expressed,

$$\frac{\sin^2 \frac{1}{2} \mathrm{I} . \cos k (\varphi - \phi' + \sigma - \sigma') \times q}{a'} \times$$

$$\frac{\tan \theta}{2} . \Big\{ b'^{(k)}_0 + b'^{(k)}_1 s' + b'^{(k)}_2 s'^2 + b'^{(k)}_3 s'^3$$

$$- \Delta b'^{(k)}_0 s - \Delta b'^{(k)}_1 2 s s' - \Delta b'^{(k)}_2 3 s s^2$$

$$+ \Delta b'^{(k)}_0 s^2 \quad + \Delta^2 b'^{(k)}_1 3 s^2 s'$$

$$- \Delta^3 b'^{(k)}_0 s^3 :$$

and the values of $b'^{(k)}_0$, $b'^{(k)}_1$, &c., will be as follow:

$$b'^{(k)}_0 = h . a^{(k)},$$

$$b'^{(k)}_1 = (- \tfrac{1}{2} + 2 k^2) \mathrm{C}^{(k)} + \frac{4 h^2 + h}{2} . \mathrm{a}^{(k)}$$

$$b'^{(k)}_2 = (3 h 2) . \frac{4 k^2 - 1}{4} . \mathrm{C}^{(k)} + \left(3 h^3 + 2 h^2 \tfrac{3}{4} h + \frac{h}{2} . k^2\right) . \mathrm{a}^{(k)}$$

$$b'^{(k)}_3 = \left\{ - \frac{24 h^2 + 27 h + 1}{24} + \frac{48 h^2 + 54 h + 1}{12} . k^2 + \frac{k^4}{3} \right\} . \mathrm{C}^{(k)}$$

$$+ \left\{ \frac{96 h^4 + 108 h^3 - 23 h^2 - 33 h}{24} + \frac{10 h^2 + 9 h}{12} k^2 \right\} . \mathrm{a}^{(k)} .$$

By taking the several orders of the finite differences of these four quantities, the other six coefficients may be numerically computed, or an independent formula may be found for calculating each of the six separately. The expansion of this part of the disturbing function is therefore reduced to the expressing of

$$\beta s'^{i'} s^{i} \cos k (\varphi - \varphi + \sigma - \sigma') \times q$$

in a series of simple cosines, i' and i being any positive numbers, zero included, the sum of which, $i' + i$, does not exceed 3, and β being the coefficient of $s'^{i'} s^{i}$ in $(s' + s)^{i' + i}$. By substituting what q stands for, we have,

$$s'^{i} s^{i} \cos k (\varphi - \varphi' + \sigma - \sigma') \times q$$

$$= \tfrac{1}{2} s'^{i'} s^{i} \cos (k - 1) (\varphi - \varphi' + \sigma - \sigma')$$

$$+ \tfrac{1}{2} s'^{i'} s^{i} \cos (k + 1) (\varphi - \varphi' + \sigma - \sigma')$$

$$- \tfrac{1}{2} s'^{i'} s^{i} \cos (\overline{k-1} . - \varphi \overline{k+1} . \varphi' - 2\nu + \overline{k-1} . \sigma - \overline{k+1} . \sigma')$$

$$- \tfrac{1}{2} s'^{i'} s^{i} \cos (k+1 . \varphi - \overline{k-1} . \varphi' - 2\nu + \overline{k+1} . \sigma - \overline{k-1} . \sigma').$$

As the arguments in the expansions of these four terms are different, each term must be computed separately, for which purpose the method in § 6 may be used. Thus

$$s'^{i'} s^{i} \cos (\overline{k-1} . \varphi - \overline{k+1} . \varphi' - 2\nu + \overline{k-1} . \sigma - \overline{k+1} . \sigma')$$

$$= \cos (\overline{k-1} . \varphi - \overline{k+1} . \varphi' - 2\nu) . \Big\{ \Big(s^{i} \cos (k-1) . \sigma\Big) \Big(s'^{i'} \cos (k+1) \sigma'\Big) + \Big(s^{i} \sin (k-1) \sigma\Big) . \Big(s'^{i'} \sin (k+1) \sigma'\Big) \Big\}$$

$$- \sin (\overline{k-1} . \varphi - \overline{k+1} . \varphi' - 2\nu) . \Big\{ \Big(s^{i} \sin (k-1) \sigma\Big) . \Big(s'^{i'} \cos (k+1) \sigma' - \Big(s^{i} \cos (k-1) \sigma\Big) . \Big(s'^{i'} \sin (k+1) \sigma'\Big) \Big\};$$

and the quantities within the brackets being known serieses, the part of the expansion of any order $e^{x} e'^{x'}$, is readily obtained.

8. One part of the disturbing function yet remains to be considered.

Expansion of $\sin^4 \tfrac{1}{2} \mathrm{I} \times \tfrac{3}{2} . \frac{r^2 r'^2}{\mathrm{D}^5} \times q^2$.

In the expression of $\frac{r r'}{\mathrm{D}^3}$ found in § 7, leave out quantities of the second and third order with respect to s and s', and we shall have,

$$\tfrac{1}{2} . \frac{r r'}{\mathrm{D}'^3} = \frac{\tan . \theta}{2} \times \Big\{ \tfrac{1}{2} \mathrm{B}'^{(0)} + \tfrac{1}{2} \mathrm{B}'^{(1)} . e' \cos \mu' - \tfrac{1}{2} \Delta \mathrm{B}'^{0} . e \cos \mu.$$

Now let

$$\tfrac{1}{2} . \frac{d \, \mathrm{B}'^{(0)}}{d (\cos \psi)} = \frac{a a'}{f} . \tfrac{3}{2} \frac{f^2}{\mathrm{D}_0^5} = \frac{a a'}{f} \times \mathrm{B}''^{(0)}$$

$$\tfrac{1}{2} . \frac{d \, \mathrm{B}'^{(1)}}{d (\cos \psi)} = \frac{a a'}{f} \Big(\tfrac{3}{4} . \frac{f^2}{\mathrm{D}_0^5} + \frac{15}{4} . \frac{f^3}{\mathrm{D}_0^7} \Big) = \frac{a a'}{f} \times \mathrm{B}''^{(1)};$$

and differentiate the formula for $\tfrac{1}{2} . \frac{r r'}{\mathrm{D}^3}$ relatively to the variable $\cos \psi$, introducing the new symbols, and observing that $\frac{a\,a'}{f} = \frac{\tan\theta}{2}$, the result will be,

$$\tfrac{3}{2} \cdot \frac{r^2 r'^2}{D^5} = \frac{\tan^2\theta}{4} \times \Big\{ B''^{(0)} + B''^{(1)} e' \cos\mu' \\ - \Delta B''^{0} e \cos\mu.$$

The quantities $B''^{(0)}$ and $B''^{(1)}$, when expressed as before in terms of $\frac{1}{D_0}$ and $\frac{f}{D_0^3}$ and their differentials, will be as follow:

$$B''^{(0)} = \tfrac{3}{2} \cdot \frac{f^2}{D_0^5} = -\tfrac{1}{2} \cdot \frac{1}{D_0} - 2\,\frac{d\,d\frac{1}{D_0}}{d\psi^2} + 2\,h \cdot \frac{f}{D_0^3},$$

$$B''^{(1)} = \tfrac{3}{4} \cdot \frac{f^2}{D_0^5} + \frac{15}{4} \cdot \frac{f^3}{D_0^7} =$$

$$-(8\,h+1) \cdot \left(\tfrac{1}{4} \cdot \frac{1}{D_0} + \frac{d\,d\frac{1}{D_0}}{d\psi^2}\right) - (8\,h^2 + h - \tfrac{9}{4}) \cdot \frac{f}{D_0^3} - \frac{d\,d\frac{f}{D_0^3}}{d\psi^2}.$$

Proceeding as before, let

$$\frac{\cos k\psi}{a'} \times b''^{(k)}_0, \text{ and } \frac{\cos k\psi}{a'} \times b''^{(k)}_2,$$

represent the expressions into which

$$B''^{(0)} \text{ and } B''^{(1)}$$

are changed, when we substitute, instead of $\frac{1}{D_0}$ and $\frac{f}{D_0^3}$ and their differentials, the parts of these quantities multiplied by $\cos k\psi$: then the part multiplied by $\cos k\psi$ in the expansion of $\sin^4 \frac{1}{2} I \times \frac{3}{2}\,\frac{r^2 r'}{D^5} \times q^2$, will be thus expressed:

$$\frac{\sin^4 \frac{1}{2} I \times \cos k(\phi - \phi' + \sigma - \sigma') \times q^2}{a'} \times$$

$$\frac{\tan^2\theta}{4} \times \Big\{ b''^{(k)}_0 + b''^{(k)}_1 \cdot e \cos\mu' \\ - \Delta b''^{(k)}_0 e \cos\mu;$$

and the values of $b''^{(k)}_0$ and $b''^{(k)}_1$ will be as follow:

$$b''^{(k)}_0 = \frac{4\,k^2 - 1}{2}\,C^{(k)} + 2\,h^2 \cdot a^{(k)},$$

$$b''^{(k)}_1 = (8\,h+1)\,\frac{4\,k^2 - 1}{4} \cdot C^{(k)} + \Big\{ 8\,h^3 + h^2 - \frac{9}{4}\,h + h\,k^2 \Big\} \cdot a^{(k)}.$$

The coefficients for any proposed value of k may now be readily computed; and nothing is wanting in this part of the problem but to reduce the factor,

$$\cos k\,(\varphi - \varphi' + \sigma - \sigma').\ \left\{\cos(\varphi - \varphi + \sigma - \sigma') - \cos(\varphi + \varphi + \sigma + \sigma' - 2\nu)\right\}^2,$$

to the form,

$$M + M' \times e \sin \mu + M'' \times e' \sin \mu',$$

and then to multiply by it, retaining in the product only quantities of the first order relative to e and e'. Now in this there is no difficulty: but the complete development of the expression would be bulky, and would contain many arguments, the greater part of which are insignificant and useless, although some of them deserve attention in particular researches. Leaving astronomers to select from the general expression the arguments which may suit their purpose, we shall here close what we had to offer respecting the development of the disturbing function, without adding to the length of this paper by any application of what we have written.

May 30, 1833.